Rutgers University: Algebra Written Qualifying Exam
January 2019: Problem 4 Solution

Exercise. Show that the groups of automorphisms of the finite abelian groups $\mathbb{Z}/30\mathbb{Z}$ and $\mathbb{Z}/15\mathbb{Z}$ are isomorphic.

Solution.

The automorphisms of $\mathbb{Z}/30\mathbb{Z}$ are $\phi_k : \mathbb{Z}/30\mathbb{Z} \to \mathbb{Z}/30\mathbb{Z}$ such that $\phi_k(g) = kg \mod 30$ where $\gcd(k, 30) = 1$, $k \in \mathbb{Z}/30\mathbb{Z}$.

Namely, ϕ_1, ϕ_7, ϕ_{11}, ϕ_{13}, ϕ_{17}, ϕ_{19}, ϕ_{23}, ϕ_{29}.

Similarly, the automorphisms of $\mathbb{Z}/15\mathbb{Z}$ are $\psi_k : \mathbb{Z}/15\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$ such that $\psi_\ell(g) = \ell g \mod 15$ where $\gcd(\ell, 15) = 1$, $\ell \in \mathbb{Z}/15\mathbb{Z}$.

Namely, ψ_1, ψ_2, ψ_4, ψ_7, ψ_8, ψ_{11}, ψ_{13}, and ψ_{14}

Let $f : Aut(\mathbb{Z}/30\mathbb{Z}) \to Aut(\mathbb{Z}/15\mathbb{Z})$ be defined by $\phi_k \mapsto \psi_{k \mod 15}$. (It isn't difficult to show that this is an isomorphism, so I am not going to write it out here.)